Superfluid phases of quark matter. II. Phenomenology and sum rules

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We derive sum rules for a uniform, isotropic superfluid quark-gluon plasma with massless quarks, first laying out the phenomenological equations obeyed by a color superconductor in terms of macroscopic observables such as the superfluid mass and baryon densities, and the electric and magnetic gluon masses, and then expressing these quantities in terms of equilibrium correlation functions. From the transverse part of the long wavelength baryon current-momentum correlation function we derive an exact expression for the superfluid baryon density, and from the longitudinal part, an *f*-sum rule. From the transverse part of the long wavelength *color* current-current correlation function we derive the superfluid Meissner mass, and from the longitudinal part, the Debye mass. These masses constrain integrals of the transverse and longitudinal parts of the gluon propagator over frequencies, and provide self-consistent conditions for a solution to the gap equation beyond weak coupling.

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Dense degenerate quark matter is expected to exhibit color superconductivity in the color-antitriplet channel; the predictions are based on a weak coupling analysis of the gap equation [1–8], as well as Ginzburg-Landau theory [2,9]. In possible physical realizations in neutron stars and ultrarelativistic heavy-ion collisions, such a superconductor would be in a strongly coupled, color-singlet state. The equilibrium properties of such superconducting matter have yet to be clarified in detail. In the strong coupling regime, Ginzburg-Landau theory delineates the possible phase diagrams near the critical temperature T_c, but quantitative predictions for the fundamental parameters of the theory are lacking.

In this paper we derive exact sum rules obeyed by the transverse and longitudinal momentum-momentum and baryon current-momentum correlation functions and the gluon propagator, D, in color superconductors. The sum rules, which are related to the linear response of the equilibrated many-body system to a current-inducing perturbation, connect the long wavelength behavior of the correlations with macroscopic observables [10,11]. Such observables include the superfluid mass density, the superfluid baryon density, and the magnetic mass or inverse penetration depth, in addition to quantities such as the charge conductivity and the Debye screening length that play a role in the normal state. These sum rules act as self-consistency conditions that must be satisfied by approximate theories of thermodynamics and correlations in the superconducting state. To derive the sum rules we first set up the phenomenological equations obeyed in a relativistic superfluid plasma in terms of the macroscopic observables, and then turn to the expressions for these observables in terms of correlation functions. We consider a uniform, isotropic color superconductor of three-flavor massless quarks at finite temperature, T, and baryon chemical potential, μ_b, and use units $\hbar = c = 1$.

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I. PHENOMENOLOGY OF RELATIVISTIC SUPERFLUIDS

Let us first review the phenomenological equations obeyed by the baryon current and momentum density of a relativistic superfluid plasma in nondissipative hydrodynamics, linearized about equilibrium with small velocities. Through these equations we derive the relativistic relation of the superfluid mass density, ρ_s, and the superfluid baryon density, n_s, quantities familiar in the nonrelativistic context [12]. Consistency of the hydrodynamic equations dictates as well the form of the superfluid acceleration equation in a relativistic superfluid. The extension of the relativistic superfluid hydrodynamic equations, Eqs. (6)–(8), (10), and (12) below, to arbitrary velocities may be found in [13].

The momentum density, $\mathbf{g}$ ($g_i = -T_{0i}$, the off-diagonal components of the stress tensor), is given in terms of the (small) velocities $\mathbf{v}_s$ of the superfluid and $\mathbf{v}_n$ of the normal components by

$$\mathbf{g} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n, \tag{1}$$

where ρ_s is the superfluid mass density, ρ_n is the normal mass density, and $\rho_s + \rho_n = \rho + P$, where ρ is the total mass density in the system at rest (the internal energy) and P is the pressure. That the superfluid velocity is an independent thermodynamic degree of freedom is a fundamental property of the paired state. In non-relativistic superfluids, $\rho + P$ reduces to mn, where m is the rest mass of the carriers, of density n; relativistically one must retain the contribution of the pressure in the momentum density. Similarly the baryon current is given in terms of the normal and superfluid velocities by

$$\mathbf{j}_b = n_s \mathbf{v}_s + n_n \mathbf{v}_n, \tag{2}$$

where n_s is the superfluid baryon density, n_n the normal baryon density, and $n_s + n_n = n_b$, the total baryon density.

The superfluid mass density and superfluid baryon density are closely related. As we derive below,

$$\rho_s = \mu_b n_s , \tag{3}$$

while the normal density obeys

$$\rho_n = \mu_b n_n + Ts, \tag{4}$$

where s is the entropy density. The latter follows from Eq. (3) together with the relation for the thermodynamic internal energy density, $\rho = \mu_b n_b + Ts - P$. Thus

$$\mathbf{g} = \mu_b \mathbf{j}_b + Ts\mathbf{v}_n . \tag{5}$$

The basic hydrodynamic equations for the superfluid are the equations of momentum and baryon conservation, and of entropy flow. In linearized hydrodynamics, **g** is driven by pressure gradients according to

$$\frac{\partial \mathbf{g}}{\partial t} + \boldsymbol{\nabla} P = 0. \tag{6}$$

Baryon conservation reads as usual,

$$\frac{\partial n_b}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}_b = 0. \tag{7}$$

Since the entropy in a superfluid system is carried only by the normal fluid, in the absence of dissipation, the entropy density obeys

$$\frac{\partial s}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{v}_n s) = 0. \tag{8}$$

Furthermore, to second order in the flow velocities, the conserved energy density, $E = T_{00}$, is given by

$$E = \frac{\rho_s}{2} v_s^2 + \frac{\rho_n}{2} v_n^2 + \rho, \tag{9}$$

and the equation for conservation of energy is

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{g} = 0. \tag{10}$$

To derive Eqs. (3) and (4), we explicitly calculate the time derivative of Eq. (9), keeping only terms of second order, and use the above equations, together with the usual first variation, $d\rho = \mu_b dn_b + Tds$, and the Gibbs-Duhem relation, $\boldsymbol{\nabla} P = n_b \boldsymbol{\nabla} \mu_b + s \boldsymbol{\nabla} T$, to find

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (\mu_b \mathbf{j}_b + Ts\mathbf{v}_n)$$

$$= (\mathbf{j}_b - n_b \mathbf{v}_n) \cdot \boldsymbol{\nabla} \mu_b + \rho_s (\mathbf{v}_s - \mathbf{v}_n) \cdot \frac{\partial \mathbf{v}_s}{\partial t}. \tag{11}$$

Identifying the energy current with the momentum density, we see that **g** is given by Eq. (5), from which Eqs. (3) and (4) follow.

In addition, the right side of Eq. (11) must vanish identically. Eliminating $\mathbf{j}_b$ there by means of Eq. (5) we find as a necessary condition for this term to vanish that $\mathbf{v}_s$ obeys the *superfluid acceleration equation*,

$$\mu_b \frac{\partial \mathbf{v}_s}{\partial t} + \boldsymbol{\nabla} \mu_b = 0. \tag{12}$$

The superfluid acceleration equation follows directly from the fact that the baryon chemical potential and superfluid velocity are given in terms of the phase ϕ of the order parameter by

$$\frac{2}{3} \mu_b = -\frac{\partial \phi}{\partial t}, \tag{13}$$

and

$$\frac{2}{3} \mu_b \mathbf{v}_s = \boldsymbol{\nabla} \phi. \tag{14}$$

The factor 2/3 is the baryon number per pair. We recall that the order parameter takes the form $\Psi_{abfh}(x) \equiv \langle \psi_{af}(x) \overline{\psi^C_{bh}}(x) \rangle = |\Psi_{abfh}(x)| e^{i\phi(x)}$, where ψ_{af} is the spinor for quarks of color a and flavor f, and $\psi^C_{af} \equiv C \overline{\psi}^T_{af}$ is the charge-conjugate spinor in the Pauli-Dirac representation. In a non-relativistic system, the first μ_b in Eq. (12) and in Eq. (14) become simply the rest mass, m, of the carriers.

An important consequence of Eq. (14) is that the circulation is quantized according to

$$\oint dl \cdot \frac{2}{3} \mu_b \mathbf{v}_s = 2\pi\nu, \tag{15}$$

where the integral is around any closed path and ν is an integer.

II. MOMENTUM AND BARYON CURRENT CORRELATION FUNCTIONS

We now derive the phenomenological superfluid densities ρ_s and n_s microscopically in terms of momentum and baryon current correlation functions, which characterize the linear response of the system to external disturbances, in particular here, a Galilean transformation. Consider the situation, following Ref. [11], in which an infinitely long cylinder containing a color superconductor moves very slowly with uniform velocity **v** along its axis, which we take to be along $\hat{z}$. We assume that the normal component is in equilibrium with the walls, so that **v** becomes the normal velocity, and that the superfluid component remains at rest.

The response of the system to this motion of the walls can be described in terms of the *transverse* response to a static long wavelength perturbation, $\int d^3\mathbf{r}\mathbf{g}(\mathbf{r}) \cdot \mathbf{v}$, where the momentum density operator is

$$g_i = \sum_{af} \psi_{af}^{\dagger}\left(-i\nabla_i\delta_{ab} - \frac{g}{2}\lambda_{ab}^{\alpha}A^{\alpha i}\right)\psi_{bf} + \sum_{\alpha} (\mathbf{E}^{\alpha}\times\mathbf{B}^{\alpha})_i ; \quad (16)$$

the A_μ^α are the color gauge fields, with field tensors $F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha - g f_{\alpha\beta\gamma} A_\mu^\beta A_\nu^\gamma$ as well as field strengths $E_i^\alpha = F_\alpha^{i0}$ and $B_i^\alpha = -\frac{1}{2}\epsilon_{ijk}F_\alpha^{jk}$, g is the color coupling constant, and the λ_{ab}^α are the Gell-Mann matrices.

The induced baryon current is given by

$$\langle \mathbf{j}_b \rangle_{\mathbf{v}} = \lim_{\mathbf{k}\to 0} \chi_T^{[jg]}(\mathbf{k},0)\mathbf{v}, \quad (17)$$

where the baryon current operator is $\mathbf{j}_b = \frac{1}{3}\Sigma_{af}\overline{\psi}_{af}\boldsymbol{\gamma}\psi_{af}$, and $\chi_T^{[jg]}(\mathbf{k},0)$ is the transverse component of the baryon current–momentum density correlation function [14],

$$\chi_{ij}^{[jg]}(\mathbf{k},z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\langle [j_{bi}, g_j]\rangle(\mathbf{k},\omega)}{z-\omega}. \quad (18)$$

We write here, for general operators $a(\mathbf{r},t)$ and $b(\mathbf{r},t)$,

$$\langle [a,b]\rangle(\mathbf{k},\omega) = -i\int d^3(\mathbf{r}-\mathbf{r}')\int_{-\infty}^{\infty} d(t-t') \times e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}e^{i\omega(t-t')}\langle [a(\mathbf{r},t), b(\mathbf{r}',t')]\rangle, \quad (19)$$

where $\langle \cdots \rangle$ is the ensemble average at given T and μ_b. The retarded commutator is given by taking the limit of z approaching the real axis from above in Eq. (18).

Comparing Eqs. (17) and (2) we see then that the normal baryon density is given in terms of the transverse baryon current–momentum density correlation function by

$$n_n = \lim_{\mathbf{k}\to 0} \chi_T^{[jg]}(\mathbf{k},0). \quad (20)$$

This equation is effectively a sum rule obeyed by $\chi_T^{[jg]}$. Similarly, the induced momentum density is given in terms of the transverse momentum density-momentum density correlation function by

$$\langle \mathbf{g} \rangle_{\mathbf{v}} = \lim_{\mathbf{k}\to 0} \chi_T^{[gg]}(\mathbf{k},0)\mathbf{v}, \quad (21)$$

where for complex frequency, z,

$$\chi_{ij}^{[gg]}(\mathbf{k},z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\langle [g_i, g_j]\rangle(\mathbf{k},\omega)}{z-\omega}. \quad (22)$$

Thus, Eq. (21) with (1) yields the transverse sum rule,

$$\lim_{\mathbf{k}\to 0} \chi_T^{[gg]}(\mathbf{k},0) = \mu_b n_n + Ts. \quad (23)$$

To derive the longitudinal versions of the sum rules (20) and (23) we suppose instead that the cylinder is still long but finite with closed ends. Then the superfluid component flows together with the normal component, leading to $\langle \mathbf{j}_b \rangle_{\mathbf{v}} = n_b \mathbf{v}$ instead of $n_n \mathbf{v}$. In this situation, the linear response analysis yields [11]

$$\langle \mathbf{j}_b \rangle_{\mathbf{v}} = \lim_{\mathbf{k}\to 0} \chi_L^{[jg]}(\mathbf{k},0)\mathbf{v}. \quad (24)$$

We thus obtain the *f*-sum rule

$$\chi_L^{[jg]}(\mathbf{k},0) = n_b, \quad (25)$$

as $|\mathbf{k}|\to 0$. Note that the *f*-sum rule (25) can be directly derived, for general $\mathbf{k}$, from the baryon conservation law (7) and the equal time commutation relation, $\langle [j_{b0}(\mathbf{r},t), \mathbf{g}(\mathbf{r}',t)]\rangle = -in_b\nabla\delta(\mathbf{r}-\mathbf{r}')$, where $j_{b0} = \frac{1}{3}\Sigma_{af}\psi_{af}^\dagger\psi_{af}$. This sum rule can be rewritten in terms of $\chi_L^{[gg]}$ as

$$\lim_{\mathbf{k}\to 0} \chi_L^{[gg]}(\mathbf{k},0) = \mu_b n_b + Ts = \rho + P. \quad (26)$$

To derive this result we calculate $\mathbf{g}$ from Eq. (5) to first order in $\mathbf{v} = \mathbf{v}_n = \mathbf{v}_s$, using $\langle \mathbf{j}_b \rangle_{\mathbf{v}} = n_b \mathbf{v}$ and the fact that the entropy term is explicitly first order in $\mathbf{v}$.

The four sum rules (20), (25), (23), and (26) relate the total and superfluid baryon densities n_b and n_s, and the long wavelength behaviors of the correlation functions $\chi_T^{[jg]}$, $\chi_L^{[jg]}$, $\chi_T^{[gg]}$, and $\chi_L^{[gg]}$. At $T=0$, as in ordinary superconductors, longitudinal first sound modes, the only low-lying excitations for colors and flavors involved in the pairing, contribute only to $\chi_L^{[jg]}$ and $\chi_L^{[gg]}$, and hence $n_s > 0$ [11].

III. COLOR PHENOMENOLOGY AND COLOR CURRENT CORRELATION FUNCTIONS

Color superconductors have the property of screening out color magnetic fields, the color Meissner effect. Following the line of argument of Ref. [11], we consider the linear response of the system to an applied static long wavelength transverse color magnetic field, $\mathbf{A}_{\text{ext}}^\gamma(\mathbf{r}) = \mathbf{A}_{\text{ext}}^\gamma(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$, where $\mathbf{k}\cdot\mathbf{A}_{\text{ext}}^\gamma(\mathbf{k}) = 0$. The external field $\mathbf{A}_{\text{ext}}^\gamma$ produces currents of various colors, β, which in turn induce color fields $\mathbf{A}_{\text{ind}}^\beta(\mathbf{r})$; the total color field is $\mathbf{A}^\beta = \mathbf{A}_{\text{ext}}^\beta + \mathbf{A}_{\text{ind}}^\beta$. In the static long wavelength limit, the induced transverse color currents are given in terms of the total transverse color field by the London equation,

$$\langle \mathbf{j}^{\alpha T}(\mathbf{r})\rangle_{\mathbf{A}_{\text{ext}}^\gamma} = -(m_M^2)_{\alpha\beta}\mathbf{A}^\beta(\mathbf{r}), \quad (27)$$

where m_M is the magnetic mass matrix, non-zero in the paired state. The inverses of its eigenvalues are the length scales on which color magnetic fields are screened in the superconductor.

To linear order in $\mathbf{A}_{\text{ext}}$, the long wavelength induced currents are given microscopically by

$$\langle \mathbf{j}^{\beta T}(\mathbf{r})\rangle_{\mathbf{A}_{\text{ext}}^\gamma} = -\lim_{\mathbf{k}\to 0} \chi_T^{\beta\gamma}(\mathbf{k},0)\mathbf{A}_{\text{ext}}^\gamma(\mathbf{r}), \quad (28)$$

where χ_T is the transverse part of the color current-current correlation function,

$$\chi_{\mu\nu}^{\alpha\beta}(\mathbf{k},z)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\langle[j_\mu^\alpha,j_\nu^\beta]\rangle(\mathbf{k},\omega)}{z-\omega},\tag{29}$$

and the color current operator for gluonic index α is

$$j_\mu^\alpha=\frac{1}{2}g\sum_{abf}\bar{\psi}_{af}\lambda_{ab}^\alpha\gamma_\mu\psi_{bf}-gf_{\alpha\beta\gamma}A^{\beta\nu}F_{\mu\nu}^\gamma.\tag{30}$$

The linearized field equation for $\mathbf{A}_{\rm ind}^\beta$,

$$\nabla\times(\nabla\times\mathbf{A}_{\rm ind}^\beta)=\langle\mathbf{j}^{\beta T}(\mathbf{r})\rangle_{\mathbf{A}_{\rm ext}^\gamma},\tag{31}$$

implies that the total color field is

$$\mathbf{A}^\beta(\mathbf{k})=[(\varepsilon^T)^{-1}]_{\beta\gamma}\mathbf{A}_{\rm ext}^\gamma(\mathbf{k}),\tag{32}$$

where $\varepsilon_{\alpha\beta}^T(\mathbf{k})$ is the static transverse color dielectric function, defined by

$$[(\varepsilon^T)^{-1}]_{\alpha\beta}(\mathbf{k})=\delta_{\alpha\beta}-\chi_T^{\alpha\beta}(\mathbf{k},0)/|\mathbf{k}|^2.\tag{33}$$

One can write $\varepsilon_{\alpha\beta}^T(\mathbf{k})$ in terms of the screened correlation function, $\tilde{\chi}_T^{\alpha\beta}(\mathbf{k},0)\equiv\chi_T^{\alpha\gamma}(\mathbf{k},0)\varepsilon_{\gamma\beta}^T(\mathbf{k})$ (the irreducible bubble), as

$$\varepsilon_{\alpha\beta}^T(\mathbf{k})=\delta_{\alpha\beta}+\tilde{\chi}_T^{\alpha\beta}(\mathbf{k},0)/|\mathbf{k}|^2.\tag{34}$$

The magnetic mass matrix is thus given in terms of $\tilde{\chi}_T$ by

$$(m_M^2)_{\alpha\beta}=\lim_{\mathbf{k}\to 0}\tilde{\chi}_T^{\alpha\beta}(\mathbf{k},0).\tag{35}$$

The transverse screening lengths are closely related to the superfluid baryon density n_s. We may see the explicit relation near T_c, where we derived the equilibrium properties utilizing general Ginzburg-Landau theory [9,15]. Within color and flavor antisymmetric pairing channels having zero total angular momentum, even parity, and aligned chirality, which include the two-flavor and color-flavor locked condensates as optimal states, we obtain the London equation for the induced current densities (28) as (see Appendix and [15])

$$\langle\mathbf{j}^{\alpha T}(\mathbf{r})\rangle_{\mathbf{A}_{\rm ext}^\gamma}=-K_T\left(\frac{g}{2}\right)^2\mathrm{Re}\{\mathrm{Tr}[((\lambda^\alpha)^*\phi_++\phi_+\lambda^\alpha)\times((\lambda^\beta)^*\phi_++\phi_+\lambda^\beta)^\dagger]_F\}\mathbf{A}^\beta(\mathbf{r})\tag{36}$$

with the coefficient

$$K_T=\frac{9n_s}{4\mu_b\mathrm{Tr}(\phi_+^\dagger\phi_+)_F}.\tag{37}$$

Here $(\phi_+)_{abfh}$ is the pairing gap of the quark of color a and flavor f with that of color b and flavor h, and the subscript F denotes the gap calculated for the paired quarks lying on the Fermi surfaces; for a color neutral system, the Fermi energies reduce to a single value $\mu_b/3$ as T approaches T_c. The relation (37) is basically that obtained by Josephson [16], $\rho_s=A_\perp(m/\hbar)^2|\Psi|^2$, for superfluid He II. Equations (27) and (36) then yield the relation between the magnetic mass, the superfluid baryon density, and the order parameter near T_c:

$$(m_M^2)_{\alpha\beta}=\frac{9g^2n_s}{16\mu_b}\frac{\mathrm{ReTr}[((\lambda^\alpha)^*\phi_++\phi_+\lambda^\alpha)((\lambda^\beta)^*\phi_++\phi_+\lambda^\beta)^\dagger]_F}{\mathrm{Tr}(\phi_+^\dagger\phi_+)_F}.\tag{38}$$

For the color-flavor locked phase, where $(\phi_+)_{abfh}=\kappa_A(\delta_{af}\delta_{bh}-\delta_{ah}\delta_{bf})$, we obtain

$$K_T=\frac{3n_s}{16\mu_b|\kappa_A|_F^2},\tag{39}$$

and

$$(m_M^2)_{\alpha\alpha}=\frac{3g^2n_s}{8\mu_b}.\tag{40}$$

Similarly, for the two-flavor channel, where $(\phi_+)_{abfh}=\epsilon_{fhs}\epsilon_{abc}d_c$ (s, the strange flavor),

$$K_T=\frac{9n_s}{16\mu_b|\mathbf{d}|_F^2},\tag{41}$$

and

$$(m_M^2)_{\alpha\alpha}=\begin{cases}0, & \alpha=1,2,3,\\ 9g^2n_s/16\mu_b, & \alpha=4,5,6,7,\\ 3g^2n_s/4\mu_b, & \alpha=8.\end{cases}\tag{42}$$

In contrast with the behavior of transverse color fields, color longitudinal fields are screened in both the normal and superconducting states. Longitudinal color charge correlations act to expel low frequency longitudinal color fields as in the response of nonrelativistic electron systems to an external longitudinal electromagnetic field [10]. We can see this behavior by simply replacing the transverse external field $\mathbf{A}_{\rm ext}^\gamma$ above with a slowly varying longitudinal one, satisfying $\mathbf{k}\times\mathbf{A}_{\rm ext}^\gamma(\mathbf{k})=0$, with time dependence $e^{-i(\omega+i\eta)t}$ (here, $\omega\simeq 0$ and η is a positive infinitesimal). Then, as in the derivation of Eq. (32), we obtain, for the total color longitudinal fields in a gauge where the scalar fields A_0^γ vanish,

$$\mathbf{A}^\beta(\mathbf{k},\omega)=[(\varepsilon^L)^{-1}]_{\beta\gamma}\mathbf{A}_{\rm ext}^\gamma(\mathbf{k},\omega),\tag{43}$$

where $\varepsilon_{\alpha\beta}^L(\mathbf{k},\omega+i\eta)\equiv\delta_{\alpha\beta}-\tilde{\chi}_L^{\alpha\beta}(\mathbf{k},\omega+i\eta)/|\mathbf{k}|^2$, with the screened correlation function $\tilde{\chi}_L^{\alpha\beta}(\mathbf{k},\omega+i\eta)=\chi_L^{\alpha\gamma}(\mathbf{k},\omega+i\eta)\varepsilon_{\gamma\beta}^L(\mathbf{k},\omega+i\eta)$, is the longitudinal color dielectric function. (Here it is more convenient to write $\tilde{\chi}_L^{\alpha\beta}\equiv\tilde{\chi}_{00}^{\alpha\beta}$, in contrast to the definition of the longitudinal part in [14].)

In the static long wavelength limit, the longitudinal correlation function reduces to minus the square of the electric mass tensor,

$$\lim_{\mathbf{k}\to 0}\tilde{\chi}_L^{\alpha\beta}(\mathbf{k},0) = -(m_E^2)_{\alpha\beta}; \tag{44}$$

then $\mathbf{A}^\alpha(\mathbf{k},0)\to|\mathbf{k}|^2(m_E^{-2})_{\alpha\beta}\mathbf{A}_{\rm ext}^\beta(\mathbf{k},0)$, so that the total longitudinal field is screened in the long wavelength limit. In the opposite limit of spatially uniformity, with slow variation in time,

$$\lim_{\omega\to 0}\lim_{\mathbf{k}\to 0}(\omega/|\mathbf{k}|)^2\tilde{\chi}_L^{\alpha\beta}(\mathbf{k},\omega+i\eta) = (\omega_p^2)_{\alpha\beta}, \tag{45}$$

where $\omega_p^{\alpha\beta}$ is the plasma frequency matrix; in this limit $\mathbf{A}^\alpha(0,\omega)\to-\omega^2(\omega_p^{-2})_{\alpha\beta}\mathbf{A}_{\rm ext}^\beta(0,\omega)$.

Near T_c, the square of the plasma frequency is given by the sum of normal and superconducting contributions, $(\omega_p^2)_{\alpha\beta}=\delta_{\alpha\beta}\omega_p^2+(m_M^2)_{\alpha\beta}$ (see Appendix); in weak coupling, $\omega_p^2=g^2\mu_b^2/18\pi^2+g^2T^2/2$ [17] and $K_T=7\zeta(3)n_b/16\pi^2T_c^2\mu_b$ [2,18], where $\zeta(3)=1.202\ldots$ is the Riemann zeta function. The deviation of $\omega_p^{\alpha\beta}$ from $\delta_{\alpha\beta}\omega_p$ below T_c comes from the fact, as clarified by Rischke *et al.* [7,19] for the two-flavor channel, that the condensate changes the color dielectric properties in such a way that the color superconductor is transparent to the low energy gluons having color charge associated with the two colors carried by a Cooper pair. In weak coupling, $m_E^{\alpha\beta}$ reduces above T_c to the usual Debye mass $\sqrt{3}\omega_p\delta_{\alpha\beta}$ [17]. Below T_c, however, $m_E^{\alpha\beta}$ generally deviates from the normal Debye mass, due to the modification of the color dielectric properties by the condensate [7,19]. Just below T_c, $(m_E^2)_{\alpha\beta}$ behaves as $3\omega_p^2\delta_{\alpha\beta}-3(m_M^2)_{\alpha\beta}$ (see Appendix).

The relations of the Meissner and Debye screening masses, m_M and m_E, to the color current-current correlation functions can be cast in terms of sum rules obeyed by the gluon propagator,

$$D_{\mu\nu}^{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = -i\langle T[A_\mu^\alpha(\mathbf{r},t)A_\nu^\beta(\mathbf{r}',t')]\rangle. \tag{46}$$

Since this propagator dominates the infrared structure of the gap equation [6], the sum rule constraints on the propagator are important to take into account in constructing a self-consistent solution to the gap equation through inclusion of polarization effects of the superconducting medium.

The transverse propagator D_T, with spectral representation,

$$D_T^{\alpha\beta}(\mathbf{k},z) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{B_T^{\alpha\beta}(\mathbf{k},\omega)}{z-\omega}, \tag{47}$$

is related to the transverse part, $\tilde{\chi}_T^{\alpha\beta}(\mathbf{k},z)$, of the irreducible current-current correlation function by

$$(D_T^{-1})_{\alpha\beta}(\mathbf{k},z) \equiv \delta_{\alpha\beta}(z^2-|\mathbf{k}|^2)-\tilde{\chi}_T^{\alpha\beta}(\mathbf{k},z). \tag{48}$$

Thus taking $z=0$ and the limit of small $|\mathbf{k}|$, we derive the transverse sum rule,

$$\lim_{\mathbf{k}\to 0}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{B_T^{\alpha\beta}(\mathbf{k},\omega)}{\omega} = (m_M^{-2})_{\alpha\beta}. \tag{49}$$

Similarly, the longitudinal propagator, $D_L=D_{00}$ in the radiation gauge, has the spectral representation,

$$D_L^{\alpha\beta}(\mathbf{k},z) = \frac{1}{|\mathbf{k}|^2}\delta_{\alpha\beta}+\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{B_L^{\alpha\beta}(\mathbf{k},\omega)}{z-\omega}, \tag{50}$$

and is given in terms of the longitudinal part of the irreducible current-current correlation function by

$$(D_L^{-1})_{\alpha\beta}(\mathbf{k},z) = \delta_{\alpha\beta}|\mathbf{k}|^2-\tilde{\chi}_L^{\alpha\beta}(\mathbf{k},z). \tag{51}$$

Again in the static long wavelength limit, we find the longitudinal sum rule:

$$\lim_{\mathbf{k}\to 0}\left(\frac{1}{|\mathbf{k}|^2}\delta_{\alpha\beta}-\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{B_L^{\alpha\beta}(\mathbf{k},\omega)}{\omega}\right) = (m_E^{-2})_{\alpha\beta}. \tag{52}$$

In the normal state the sum rules (49) and (52) reproduce the results analyzed by Pisarski and Rischke [6] in terms of the spectral representation up to one-loop order. The Meissner masses vanish in the transverse sector; Landau diamagnetism leads only to $|\mathbf{k}|^2$ corrections, and not a nonzero screening mass in the limit $\mathbf{k}\to 0$. The right side of Eq. (49) is replaced by $\delta_{\alpha\beta}\lim_{\mathbf{k}\to 0}|\mathbf{k}|^{-2}$ to leading order in g, as in Ref. [6]. The longitudinal sector contains the usual Debye screening, which is characterized by $(m_E^2)_{\alpha\beta}=3\omega_p^2\delta_{\alpha\beta}$ to leading order in g.

Rischke [7] explicitly calculated the zero temperature, low frequency, long wavelength limit of the color current-current correlation function for the two-flavor and color-flavor locked condensates to leading order in g. Repeating his calculations for three flavors, we find the screening masses (with no sum over α),

$$(m_E^2)_{\alpha\alpha} = 3(m_M^2)_{\alpha\alpha} = \frac{21-8\ln 2}{18}\omega_p^2, \quad \alpha=1,\ldots,8 \tag{53}$$

for color-flavor locking, and

$$(m_E^2)_{\alpha\alpha}=\omega_p^2, \quad (m_M^2)_{\alpha\alpha}=0, \quad \alpha=1,2,3,$$

$$(m_E^2)_{\alpha\alpha}=2\omega_p^2, \quad (m_M^2)_{\alpha\alpha}=\omega_p^2/3, \quad \alpha=4,5,6,7, \tag{54}$$

$$(m_E^2)_{88}=3\omega_p^2, \quad (m_M^2)_{88}=2\omega_p^2/9$$

for the two-flavor channel in which only quarks of color *R* and *G*, which couple to gluons of $\alpha=1,2,3$, undergo BCS pairing.[1] (The choice of the two colors involved in the pairing is arbitrary under the constraint of overall color neutrality.) The screening masses depend on color charge only in the two flavor condensate, because there the order parameter is *anisotropic* in color space in contrast to that in the color-flavor locked condensate. How static transverse screening influences the pairing gap compared with Landau damping of color magnetic gluons [20] depends sensitively on the gluon energies and momenta controlling the pairing gap [7].

Weak coupling calculations ignoring the effects of the superconducting medium yield the logarithm of the gap [4–6],

$$\ln(\Delta/\mu_b) = -3\pi^2/\sqrt{2}g - 5\ln g + \cdots. \tag{55}$$

The weak coupling solution is not self-consistent in the sense that it satisfies the sum rules (49) and (52) with the masses $m_E^{\alpha\beta} = \sqrt{3}\omega_p\delta_{\alpha\beta}$ and $m_M^{\alpha\beta} = 0$, corresponding to the thermodynamics of the weakly interacting normal gas [6]. As Rischke [8] showed to one loop order, the superconducting medium significantly modifies the gluon self-energy from the normal medium value only in the energy range $|\omega| \lesssim \Delta$, which is not sufficient to change the logarithm of the gap from Eq. (55) up to subleading order in g. On the other hand, a self-consistent solution that satisfies the sum rules (49) and (52) with masses given by Eqs. (53) and (54) would include contributions in all orders. The sum rules thus provide a check on approximate theories for the pairing gap beyond weak coupling.

In summary, we have derived the transverse and longitudinal sum rules for a color superconductor that is uniform and isotropic in ordinary space. In doing so we have brought out the relation of the long wavelength behaviors of the momentum-momentum, baryon current-momentum, and color current correlation functions to macroscopic quantities such as the superfluid density and the Meissner and Debye masses, as well as the relevance of the sum rules for the gluon propagator to the self-consistent solution to the gap equation. The sum rules hold for any number of flavors and quark masses. It is straightforward to extend the present analysis for color current-current correlations to the case in which the electric currents coexist with the color currents; in this situation the electric currents modify the supercurrents through a mixing of color and electric charge (see, e.g., Refs. [15,21]).

[1]The Debye and Meissner masses (54), calculated for a three-flavor system, are different from those obtained by Rischke [7] for a two-flavor system. This difference stems from the fact that in a three-flavor system the third flavor (e.g., strange), while not participating in pairing, contributes to the Debye masses $(m_E^2)_{\alpha\alpha}$; the square of the normal plasma frequency ω_p^2 is proportional to N_f, the number of flavors in the system. The masses (54) can thus be derived by multiplying the Debye and Meissner masses given in Table I of the first paper in Ref. [7] by a factor 2/3 due to N_f and adding to the resultant Debye masses the third-flavor contribution $\delta_{\alpha\beta}\omega_p^2$.

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APPENDIX: THE GINZBURG-LANDAU REGION

In this appendix, we summarize the derivation of the London equation (36) near T_c, for color and flavor antisymmetric pairing channels having zero total angular momentum, even parity, and aligned chirality; full details will be given in Ref. [15]. In the presence of applied weak color fields, $A_{\rm ext}^{\alpha\mu}(x)$, of very long wavelength and low frequency, the resultant gradient of the order parameter adds a small correction to the homogeneous part of the Ginzburg-Landau free energy derived in Ref. [9]. Up to second order in the gap, this energy correction is

$$\Omega_g = \frac{1}{2}K_T \mathrm{Tr}[(D_i\phi_+)^\dagger D_i\phi_+]_F + \frac{1}{2}K_L \mathrm{Tr}[(D_0\phi_+)^\dagger D_0\phi_+]_F, \tag{A1}$$

where the covariant derivative is $D_\mu\phi_+ \equiv \partial_\mu\phi_+ - \frac{1}{2}ig[(\lambda^\alpha)^*\phi_+ + \phi_+\lambda^\alpha]A_\mu^\alpha$ with the total color fields A_μ^α. The time dependence of the gap is, by definition, measured with respect to that in the equilibrium phase, i.e., with respect to the phase factor $e^{-2i\mu_b t/3}$ arising in the order parameter from Eq. (13) (see Ref. [9]). As we show here, the coefficient K_T is given by Eq. (37). The coefficient K_L is not necessarily equal to K_T since Lorentz invariance is broken in a many particle system. In weak coupling, K_L reduces to $3K_T$ [7]. As required, Ω_g is invariant under global $U(1)$ gauge transformations and flavor rotations, as well as under local color $SU(3)$ gauge transformations.

To derive the dependence of the coefficient K_T on the superfluid baryon density n_s, Eq. (37), we consider the situation in the absence of color fields, in which the pairs move uniformly with small constant velocity $\mathbf{v}_s$ and the normal fluid remains at rest. The phase factor of the gap in the fixed frame transforms by $\phi_+ \to e^{i\mathbf{P}\cdot\mathbf{r}}\phi_+$, where $\mathbf{P}$ is the total pair momentum. The total momentum of the superfluid is then

$$3n_s\mathbf{P}/2 = \rho_s\mathbf{v}_s. \tag{A2}$$

In this situation,

$$\Omega_g = \frac{1}{2}K_T\mathbf{P}^2\mathrm{Tr}(\phi_+\phi_+^\dagger)_F. \tag{A3}$$

Following de Gennes [22] and Wölfle [23], we then obtain the baryon current density from the usual canonical equation for the gradient energy density (A3) as

$$\mathbf{j}_s = \frac{2}{3}\frac{\delta\Omega_g}{\delta\mathbf{P}} = \frac{2}{3}K_T\mathrm{Tr}(\phi_+^\dagger\phi_+)_F\mathbf{P}. \tag{A4}$$

Using Eq. (A2) to eliminate **P**, and Eqs. (2) and (3), we derive Eq. (37). Note that the extra time variation of the gap leads to terms in Ω_g of order $\mathbf{P}^4$, which play no role here.

The color current densities induced by the applied weak color fields near T_c are composed of the superfluid and normal contributions, $j_s^{\alpha\mu}$ and $j_n^{\alpha\mu}$. The superfluid color currents can be calculated as

$$\begin{aligned} j_s^{\alpha\mu} &= \frac{\delta\Omega_g}{\delta A_\mu^\alpha} \\ &= -[K_T+\delta_{\mu 0}(K_L-K_T)]\frac{g}{2}\mathrm{Im}\{\mathrm{Tr}[((\lambda^\alpha)^*\phi_+ \\ &\quad +\phi_+\lambda^\alpha)^\dagger\partial_\mu\phi_+]_F\}+[K_T+\delta_{\mu 0}(K_L-K_T)]\left(\frac{g}{2}\right)^2 A_\mu^\beta \\ &\quad \times\mathrm{Re}\{\mathrm{Tr}[((\lambda^\alpha)^*\phi_+ +\phi_+\lambda^\alpha)((\lambda^\beta)^*\phi_+ +\phi_+\lambda^\beta)^\dagger]_F\}. \end{aligned} \tag{A5}$$

Equation (A5) reduces to the London equation (36) for applied static transverse color fields when the spatial variation is of sufficiently long wavelength that we can ignore the term containing $\partial_\mu\phi_+$.

We conclude this appendix by considering the linear response to applied longitudinal color fields near T_c. For static color fields of very long wavelength the induced color density $j_s^{\alpha 0}$ can be obtained from Eq. (A5) as $j_s^{\alpha 0} =(K_L/K_T)(m_M^2)_{\alpha\beta}A^{\beta 0}$ where m_M^2 is given by Eq. (38). In weak coupling, this color density, together with the normal contribution $-3\omega_p^2 A^{\alpha 0}$, leads to the square of the Debye screening mass matrix, $(m_E^2)_{\alpha\beta}=3\omega_p^2\delta_{\alpha\beta}-3(m_M^2)_{\alpha\beta}$. For uniform color fields varying very slowly in time, we calculate the induced supercurrent from Eq. (A5) as $\mathbf{j}_s^\alpha= -(m_M^2)_{\alpha\beta}\mathbf{A}^\beta$, where m_M^2 is again given by Eq. (38). Combining this supercurrent with the induced normal current $-\omega_p^2\mathbf{A}^\alpha$, we obtain the square of the plasma frequency matrix, $(\omega_p^2)_{\alpha\beta}=\delta_{\alpha\beta}\omega_p^2+(m_M^2)_{\alpha\beta}$.

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